Chapter 2: Periodic Motion

1. Show that the total energy of the particles executing SHM is constant. [Asked in: 2082, 2081, 2080, 2076, 2073 | Very High Probability]

Solution

When a body is executing SHM, it possesses both kinetic and potential energy. K.E. is due to its motion and P.E. is due to restoring force that tends to bring the body back to mean position.

Suppose a particle of mass 'm' is executing SHM about its mean position. Let, r be the amplitude of the motion and y be the displacement of the body. The velocity of the body is:

v = $\omega\sqrt{r^2 - y^2}$ .... (i)

KE = $\frac{1}{2}mv^2$ = $\frac{1}{2}m\omega^2(r^2 - y^2)$ .... (ii)

Small workdone by the body while displacing it by a small distance dy is given by

dw = -Fdy, where, -ve sign shows that F and dy are oppositely directed.

Then, total work done is:

$\int_0^y dw = \int_0^y -Fdy$

= $-\int_0^y (ma)\, dy$

= $-\int_0^y m(-\omega^2 y)\, dy$ [∵ a = $-\omega^2$y in SHM]

= $m\omega^2 \left[\frac{y^2}{2}\right]$ = $\frac{1}{2}m\omega^2 y^2$

This workdone is stored as P.E.

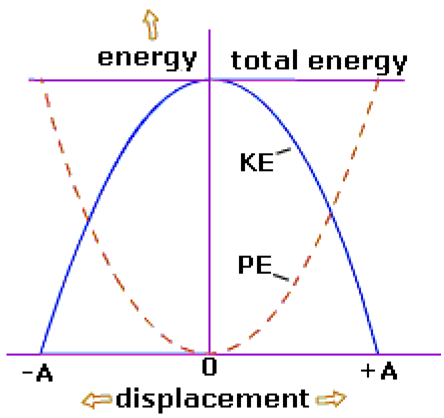
∴ P.E. = $\frac{1}{2}m\omega^2 y^2$ .... (iii)

The total energy is given by:

TE = KE + PE = $\frac{1}{2}m\omega^2(r^2 - y^2) + \frac{1}{2}m\omega^2 y^2$

∴ TE = $\frac{1}{2}m\omega^2 r^2$ .... (iv)

The eq$^{n.}$ (iv) shows that the total energy of a particle executing SHM remains constant throughout the motion. The variation of K.E., P.E. and T.E. with displacement is given in the figure below:



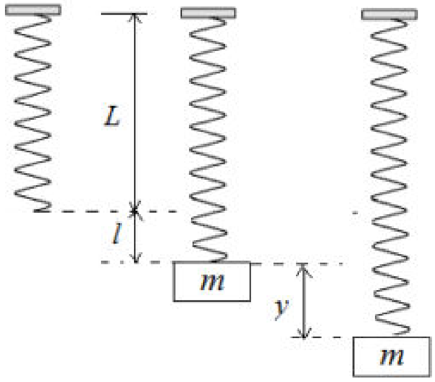
2. Obtain an expression for frequency of oscillation of vertical mass spring system. [Asked in: 2081, 2073, 2072 | Medium Probability]

Solution

Let, a small body of mass 'm' be suspended by a light helical spring whose one end is fixed to a rigid support as shown in figure. Due to the weight of the body, the spring stretches by extension l, then, restoring force acting on it is given by,

$F_1 = -kl$

Now, the body is pulled down vertically so that it is further displaced by 'y' then, the new restoring force offered by the spring is



$F_2 = -k(l + y)$

So, net force on the system is given by,

$F = F_2 - F_1$

$= -k(l + y) - (-kl) = -ky$

∴ ma = -ky [∵ F = ma]

Or, a = $-\frac{k}{m}y$

Since, k and m are constants,

a ∝ - y.

Hence, the motion of the mass attached to the spring in the horizontal plane is SHM.

Time period

Comparing a = $-\frac{k}{m}y$ with a = $-\omega^2 y$.

$\omega^2 = \frac{k}{m}$

Or, $\omega = \sqrt{\frac{k}{m}}$

Or, $\frac{2\pi}{T} = \sqrt{\frac{k}{m}}$

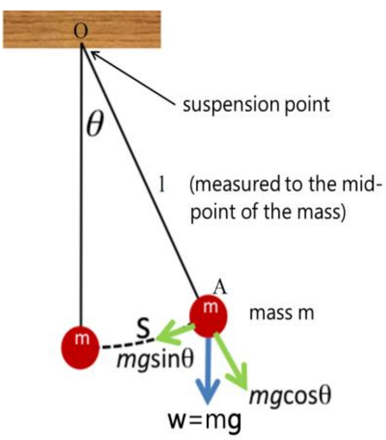
∴ $T = 2\pi\sqrt{\frac{m}{k}}$

This is the expression for time – period.

3. **Show that motion of a simple pendulum is simple harmonic and hence calculate its time period.**
[Asked in: 2081, 2080, 2079, 2072 | Extreme Probability]

Solution

A simple pendulum is a system of a point mass (bob) suspended at the end of a weightless and inextensible thread attached to a rigid support. The length from the point of suspension to the centre of gravity (cg) of the bob is called its effective length as shown in figure.



Let us consider, a bob of mass m is suspended to a string of effective length (l). It is displaced by angular displacement θ, then, the weight of the bob (mg) can be resolved into two components:

(i) mgcosθ, that provides the tension (T) in the string.

(ii) $mgsin\theta$, that provides necessary restoring force to the bob. i.e.

Restoring force (F) = -mgSinθ

or, ma = -mgSinθ

Or, a = -gSinθ .... (i)

If θ is small, then, Sinθ ≈ θ

From eq$^{n.}$ (i), a = -gθ

Or, a = -g$\frac{y}{l}$ .... (ii)

Equation (ii) shows that the acceleration is directly proportional to displacement and directed towards mean position. Hence, motion of simple pendulum is S.H.M.

Again, to find time period,

a = - ω^2y = - g$\frac{y}{l}$

Or, $\left(\frac{2\pi}{T}\right)^2 = \frac{g}{l}$

∴ T = 2π $\sqrt{\frac{l}{g}}$

And, **f = $\frac{1}{T}$ = $\frac{1}{2\pi}\sqrt{\frac{g}{l}}$**