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## Chapter 2: Periodic Motion

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### Periodic motion:

The motion that repeats in the regular (equal) interval of time is called periodic motion. Periodic motion can be circular, elliptical, Oscillatory etc.

For example, elliptical  $\rightarrow$  motion of earth.

Oscillatory  $\rightarrow$  Motion of pendulum, motion of swing, motion of air particles in sound wave, motion of tuning fork, motion of guitar wire, motion of water particles when stone is dropped on water.

Circular  $\rightarrow$  a stone tied on the thread and rotated, motion of cycle in curve path etc.

### Oscillatory motion or vibratory motion:

To and fro motion about the mean position is called oscillatory or vibratory motion. Small oscillations are called vibration. Here, amplitude is small.

### Harmonic motion:

If the motion of any object like displacement, velocity, acceleration etc. can be expressed in terms of harmonic functions like sine, cosine etc. then, the motion is called harmonic motion.

### Simple Harmonic motion:

The harmonic motion of simplest type (i.e. amplitude, frequency, period) is called simple harmonic motion.

It deals with an oscillatory motion (to and fro) in which the acceleration produced by the body is directly proportional to the displacement produced & acceleration is always directed towards a fixed point called mean position. i.e.,  $a \propto y$ , where  $a$  is acceleration &  $y$  is displacement.

Or,  $a = -ky$ .

Negative sign shows that  $a$  and  $y$  are in opposite direction.

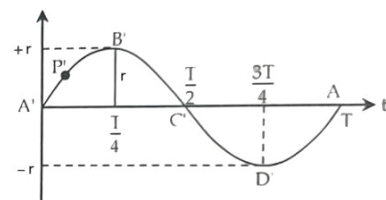
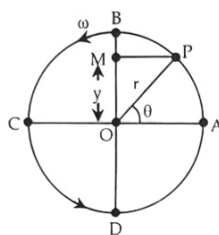
Note: If a guitar wire is plucked then,

It moves back and forth.

If a paper boat is kept in still water, and if a stone is dropped, paper boat (water particles) moves back and forth i.e. simple harmonic motion.

### Equation of Simple Harmonic Motion:

The motion of a particle in a circle is periodic but not oscillatory. But, if we project the motion of the particle to any linear path, the motion becomes periodic as well as oscillatory.



Let us consider, a particle is moving around a circle of radius  $r$  with uniform angular velocity  $\omega$ . Let, at any instant  $t$ ,  $P(x, y)$  be the position of the particle and  $\theta$  be the angle at origin. The projection  $OM$  on

y-axis from the point P represents the displacement of the particle at time t, as shown in figure along with the graph.

**Displacement (y):**

In the figure above, OM is the displacement. In  $\Delta OPM$ ,  $\sin \angle OPM = \sin \theta = \frac{OM}{OP}$   

$$= \frac{y}{r}$$

Or,  $y = r \sin \theta$  ..... (i)

Now, angular displacement is  $\theta$  for which angular velocity ' $\omega$ ' =  $\frac{\theta}{t}$

Or,  $\theta = \omega t$ .

Putting the value of  $\theta$  in eq<sup>n</sup>. (i), we get,

$y = r \sin \omega t$

**Velocity (v):**

It is the rate of change of displacement. It is denoted by v. i.e.,  $v = \frac{dy}{dt}$

We have,  $y = r \sin \omega t$  where, amplitude (A) = r

Or,  $\frac{dy}{dt} = \frac{d}{dt} (r \sin \omega t)$   

$$= r \frac{d(\sin \omega t)}{d(\omega t)} \times \frac{d(\omega t)}{dt}$$

Or,  $v = r \omega \cos \omega t$

Or,  $v = r \omega \sqrt{1 - \sin^2 \omega t}$

Or,  $v = \omega \sqrt{r^2 - r^2 \sin^2 \omega t}$

Or,  $v = \omega \sqrt{r^2 - y^2}$

i) At mean position,  $y = 0$ .

So,  $v = \omega \sqrt{r^2 - y^2}$

Or,  $v = \omega \sqrt{r^2 - 0}$

Or,  $v = \omega r$ .

So, velocity is maximum at mean position.

ii) At extreme position,  $y = r$ .

So,  $v = \omega \sqrt{r^2 - y^2}$

Or,  $v = \omega \sqrt{r^2 - r^2}$

Or,  $v = 0$ .

So, velocity is minimum (zero) at extreme position.

**Acceleration:**

The rate of change of velocity is called acceleration. It is denoted by a.

i.e.,  $a = \frac{dv}{dt}$

Or,  $a = \frac{d}{dt} (r \omega \cos \omega t)$

$$= r\omega \frac{d(\cos\omega t)}{d\omega} \times \frac{d(\omega t)}{dt}$$

$$= -r\omega \sin\omega t \cdot \omega$$

$$\text{Or, } a = -r\omega^2 \sin\omega t$$

$$\therefore a = -\omega^2 y$$

i) At mean position,  $y = 0$ .

$$a = -\omega^2 y = \omega^2 \cdot 0 = 0.$$

i.e., acceleration produced in mean position is zero.

ii) At extreme position,  $y = r$ .

$$\text{So, } a = -\omega^2 r$$

### **Time Period:**

It is the time required for one complete oscillation. It is denoted by T. We have, the magnitude of acceleration of a body executing S.H.M. is:

$$a = -\omega^2 y$$

$$a = \left(\frac{2\pi}{T}\right)^2 y$$

$$\text{Or, } a = \frac{4\pi^2}{T^2} y$$

$$\text{Or, } T^2 = \frac{4\pi^2}{a} y$$

$$\therefore T = 2\pi \sqrt{\frac{y}{a}}$$

### **Frequency:**

It is no. of oscillation in one second.

$$f = 1/T.$$

$$= \frac{1}{2\pi \sqrt{\frac{y}{a}}} = \frac{1}{2\pi} \sqrt{\frac{\text{Acceleration (a)}}{\text{Displacement (y)}}}$$

### **Phase ( $\phi$ ):**

Phase explains the state of particle at any instant of time. The state of particle means where is the particle and what is the direction of motion of the particle.

In fact, phase is the angle (angular displacement) which is represented by  $\phi$ . Its unit is radian.

$$\phi = \frac{2\pi}{\lambda} x \text{ where, } x = \text{displacement.}$$

### **Simple Harmonic Motion:**

This is an oscillatory motion where an acceleration of a body is directly proportional to displacement and acceleration is always directed towards fixed point. So, if a particle moves in 'to and fro' motion in a straight line such that acceleration produced is directly proportional to the displacement from that position, it is said to have S.H.M.

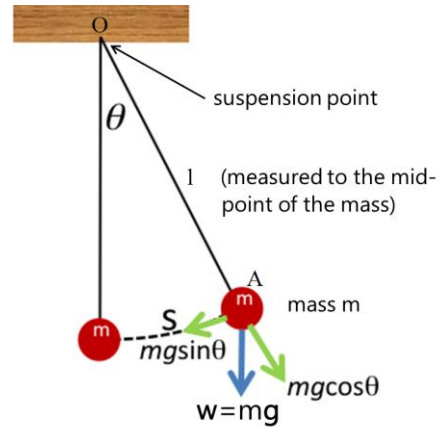
### **Characteristics of S.H.M.**

For a body executing S. H.M

- (i) It is to and fro motion about their mean position.
- (ii) Its acceleration is directly proportional to displacement from mean position. i.e.,  $a \propto y$
- (iii) Its acceleration always directed towards mean position.
- (iv) It is periodic motion.

**Simple Pendulum:**

A simple pendulum is a system of a point mass (bob) suspended at the end of a weightless and inextensible thread attached to a rigid support. The length from the point of suspension to the centre of gravity (cg) of the bob is called its effective length as shown in figure.



Let us consider, a bob of mass  $m$  is suspended to a string of effective length ( $l$ ). It is displaced by angular displacement  $\theta$ , then, the weight of the bob ( $mg$ ) can be resolved into two components:

- (i)  $mg \cos \theta$ , that provides the tension ( $T$ ) in the string.
- (ii)  $mg \sin \theta$ , that provides necessary restoring force to the bob. i.e.

Restoring force ( $F$ ) =  $-mg \sin \theta$

or,  $ma = -mg \sin \theta$

Or,  $a = -g \sin \theta$  ..... (i)

If  $\theta$  is small, then,  $\sin \theta \approx \theta$

From eq<sup>n</sup>. (i),  $a = -g \theta$

Or,  $a = -g \frac{y}{l}$  ..... (ii)

Equation (ii) shows that the acceleration is directly proportional to displacement and directed towards mean position. Hence, motion of simple pendulum is S.H.M.

Again, to find time period,

$a = -\omega^2 y = -g \frac{y}{l}$

Or,  $\left(\frac{2\pi}{T}\right)^2 = \frac{g}{l}$

$\therefore T = 2\pi \sqrt{\frac{l}{g}}$

And,  $f = \frac{1}{T} = \frac{1}{2\pi} \sqrt{\frac{g}{l}}$

**Laws of simple pendulum**

1. Time period of Simple Pendulum (S.P.) is independent of mass of bob.
2. Time period of S.P. is directly proportional to amplitude.
3. Time period of S.P. is directly proportional to root of length of pendulum i.e.,  $T \propto \sqrt{l}$ .

4. Time period of S.P. is inversely proportional to root of g i.e.,  $T \propto \sqrt{\frac{l}{g}}$

### Second's pendulum

Pendulum whose time period is 2 second is called second's pendulum.

From  $T = 2\pi \sqrt{\frac{l}{g}}$ , if  $T = 2\text{Sec.}$ ,  $l = 99.3 \text{ cm.}$

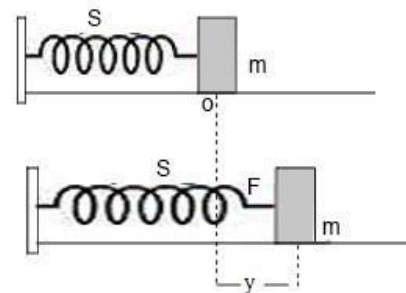
### Drawbacks of simple pendulum

1. Point but heavy mass is ideal case.
  2. Weightless and inextensible thread is ideal case.
  3. Perfectly rigid support is also ideal case.
  4.  $T = 2\pi \sqrt{\frac{l}{g}}$  it is true only for a small  $\theta$  (angular distance,  $\theta \leq 4^\circ$ ).
- If  $\theta$  is large, motion can't be linear.
5. Presence of air resistance is not negligible but here it is not included.

### Loaded mass spring (Horizontal Spring):

Let us consider, a spring whose one end is connected to rigid support and another end is connected to mass  $m$ . Spring is weightless (light) and it is kept on a horizontal frictionless table.

Initially, mass ( $m$ ) is at position  $o$  i.e., mean position. If the body is displaced to "A" by displacement  $y$  with applied force and then released, then it moves in back-and-forth direction.



A mass oscillating on light stretched spring

From Hook's law, restoring force ( $A \propto \text{extension } (y)$ ) i.e.

$F = -ky$  where  $k$  is proportionality constant or spring constant.

Or,  $ma = -ky$ .

$$\therefore a = -\frac{k}{m}y.$$

This shows that acceleration is directly proportional to displacement and directed towards mean position. Hence, motion is S.H.M.

$$\text{Also, } a = -\omega^2 y = -\frac{k}{m}y$$

$$\therefore T = 2\pi \sqrt{\frac{m}{k}}$$

$$\text{And, } f = \frac{1}{T} = \frac{1}{2\pi} \sqrt{\frac{k}{m}}$$

### Loaded mass spring (vertical spring)

Let, a small body of mass ' $m$ ' be suspended by a light helical spring whose one end is fixed to a rigid support as shown in figure. Due to the weight of the body, the spring stretches by extension  $l$ , then, restoring force acting on it is given by,

$$F_1 = -kl$$

Now, the body is pulled down vertically so that it is further displaced by 'y' then, the new restoring force offered by the spring is

$$F_2 = -k(l + y)$$

So, net force on the system is given by,

$$F = F_2 - F_1$$

$$= -k(l + y) - (-kl) = -ky$$

$$\therefore ma = -ky \quad [\because F = ma]$$

$$\text{Or, } a = -\frac{k}{m}y$$

Since, k and m are constants,

$$a \propto -y.$$

Hence, the motion of the mass attached to the spring in the horizontal plane is SHM.

### Time period

Comparing  $a = -\frac{k}{m}y$  with  $a = -\omega^2y$ .

$$\omega^2 = \frac{k}{m}$$

$$\text{Or, } \omega = \sqrt{\frac{k}{m}}$$

$$\text{Or, } \frac{2\pi}{T} = \sqrt{\frac{k}{m}}$$

$$\therefore T = 2\pi \sqrt{\frac{m}{k}}$$

This is the expression for time – period.

### Energy in S.H.M.:

When a body is executing SHM, it possesses both kinetic and potential energy. K.E. is due to its motion and P.E. is due to restoring force that tends to bring the body back to mean position.

Suppose a particle of mass 'm' is executing SHM about its mean position. Let, r be the amplitude of the motion and y be the displacement of the body. The velocity of the body is:

$$v = \omega\sqrt{r^2 - y^2} \dots\dots\dots (i)$$

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2(r^2 - y^2) \dots\dots\dots (ii)$$

Small workdone by the body while displacing it by a small distance dy is given by

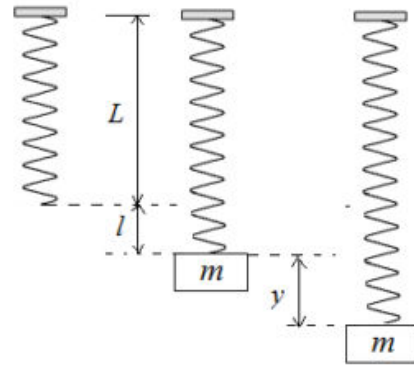
$dw = -Fdy$ , where, -ve sign shows that F and dy are oppositely directed.

Then, total work done is:

$$\int_0^y dw = \int_0^y -Fdy$$

$$= -\int_0^y (ma) dy$$

$$= -\int_0^y m(-\omega^2y) dy \quad [\because a = -\omega^2y \text{ in SHM}]$$



$$= m\omega^2 \left[ \frac{y^2}{2} \right] = \frac{1}{2} m\omega^2 y^2$$

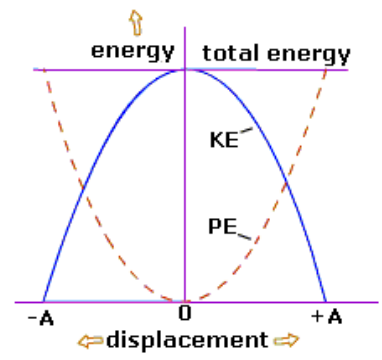
This workdone is stored as P.E.

$$\therefore \text{P.E.} = \frac{1}{2} m\omega^2 y^2 \dots\dots\dots (iii)$$

The total energy is given by:

$$\text{TE} = \text{KE} + \text{PE} = \frac{1}{2} m\omega^2 (r^2 - y^2) + \frac{1}{2} m\omega^2 y^2$$

$$\therefore \text{TE} = \frac{1}{2} m\omega^2 r^2 \dots\dots\dots (iv)$$

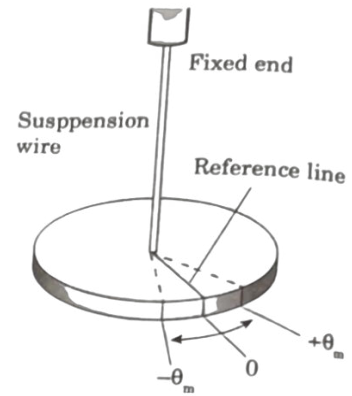


The eq<sup>n</sup> (iv) shows that the total energy of a particle executing SHM remains constant throughout the motion. The variation of K.E., P.E. and T.E. with displacement is given in the figure below:

**Angular Simple Harmonic Motion:**

Angular SHM is defined as the oscillatory motion of a body in which the torque ( $\tau$ ) is directly proportional to the angular displacement ( $\theta$ ) and its direction is opposite to that of angular displacement.

Figure alongside shows a torsion pendulum which consists of a horizontal disc suspended on a wire. It is angular version of a simple harmonic oscillator. If the disc is rotated by some angular displacement  $\theta$  from its rest position ( $\theta = 0$ ) and release it, it will oscillate about that position in angular SHM. The restoring torque developed on it is



$$\tau = -k\theta, \text{ where, } k = \text{a constant called torsion constant.}$$

As we know,  $\tau = I\alpha$ , where,  $I$  is moment of inertia of the disc

$$\text{So, } I\alpha = -k\theta$$

$$\text{Or, } \alpha = -\frac{k}{I}\theta$$

Since,  $k$  &  $I$  are constant,  $\alpha \propto -\theta$ . This shows that, the angular acceleration is directly proportional to angular displacement and directed opposite to it. So, motion of the disc is angular SHM.

Now,

$$\alpha = -\omega^2\theta$$

$$\text{Or, } -\frac{k}{I}\theta = -\omega^2\theta$$

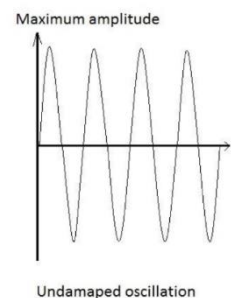
$$\text{Or, } \left(\frac{2\pi}{T}\right)^2 = \frac{k}{I}$$

$$\text{Or, } T = 2\pi\sqrt{\frac{I}{k}}$$

This is the required relation for time period of angular SHM.

**Free oscillation or Undamped Oscillation:**

When a body is capable of oscillating without external resistive forces, then, it vibrates or oscillates with its natural frequency. This type of oscillation is called free oscillation. Eg.: oscillation of simple pendulum in vacuum.



### Damped Oscillation:

The oscillation, in which amplitude of oscillation gradually decreases and becomes zero due to external resistive force is called damped oscillation. Eg.: motion of simple pendulum in air.

### Forced Oscillation:

When a body is compelled to oscillate by an external periodic force with its frequency other than the natural frequency of the body. Then, it is called forced oscillation. Eg. vibration of air column of Resonance apparatus.

### Resonance:

It is a special type of forced oscillation in which frequency of external periodic force is equal to natural frequency of body. In this condition, the body vibrates with maximum amplitude.

Example:

- i. Sometimes, glass of windows of a house are found to be broken when large sounding airplanes passes by.
- ii. While soldiers cross a bridge, they are ordered to break their steps. If they do not do so, then bridge may start to oscillate and by any change if frequency of their footsteps matches with natural frequency of bridge, then, the bridge may collapse.

